Some remarks on analytic geometric acoustics of the vocal tract.

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Symbols:

p : soundpressure

ρ : density of the air

ρ_0 : density of the air at rest

v : velocity of an air particle in the x-direction

c : velocity of sound in air

ϕ : velocity potential

σ : cross-section of the vocal tract

The vocal tract can be considered as a tube with a cross-section which is time- and place-dependent. Figure 1 shows a part of such a tube at the moments t_o and $t_o + \Delta t$. Δt and Δx are to be assumed infinitesimal.

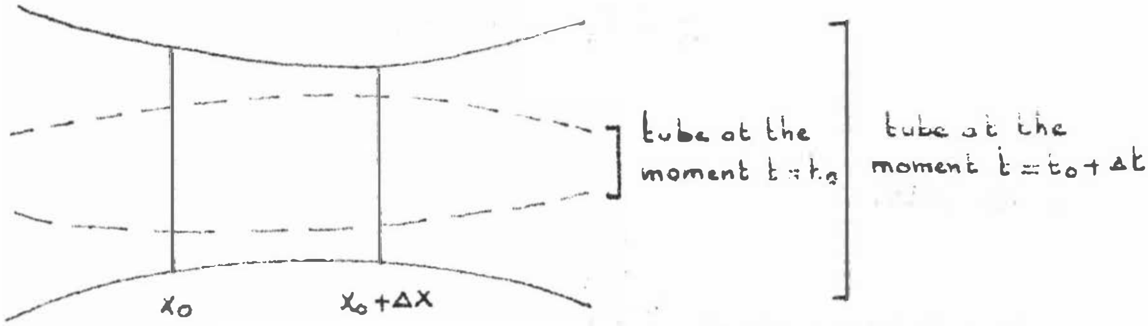


fig. 1

Application of $F = \frac{d}{dt}(mv)$ on a "moving" volume part, enclosed by the cross-sections at x_o and $x_o + \Delta x$, leads, after taking the limit Δt, $\Delta x \to o$ and some simple approximations, to the expression:

$$(1) \qquad -\sigma \frac{\partial p}{\partial x} = \rho_o \frac{\partial}{\partial t}(\sigma v) \; .$$

Considerations of continuity applied on a fixed part of the tube, enclosed by the cross-sections at x_o and x_o + Δx, give us an equation of continuity:

$$(2) \qquad -\rho_o \frac{\partial}{\partial x}(\sigma v) = \frac{\partial}{\partial t}(\rho \sigma) \; .$$

Using the adiabatic law:

$$(3) \qquad p = c^2 \rho \qquad ,$$

we obtain after introducing a velocity potential:

$$(4) \quad \Phi(x,t) \text{ with } v = -\frac{\partial \Phi}{\partial x} \text{ and } p = \rho_o \frac{\partial \Phi}{\partial t}$$

from the equations (1) to (4) inclusive the following second order partial differential equations:

$$(5) \qquad \frac{\partial}{\partial x}\left(\sigma \frac{\partial \Phi}{\partial x}\right) = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\sigma \frac{\partial \Phi}{\partial t}\right)$$

$$(6) \qquad \frac{\partial}{\partial x}\left(\sigma \frac{\partial p}{\partial x}\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\sigma\, p\right) .$$

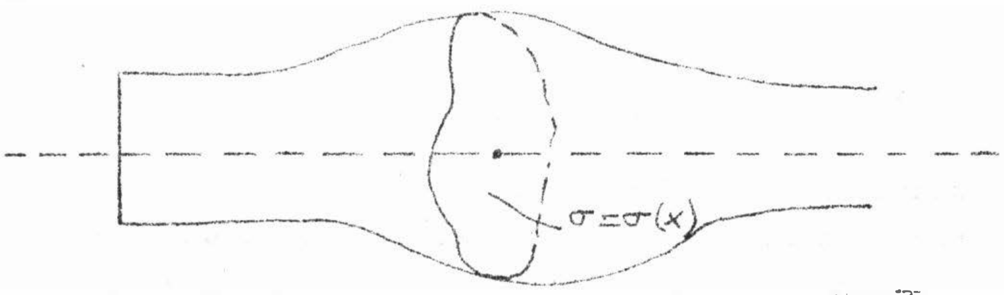
The equations (5) and (6) give a description of the acoustical behaviour of a tube with a cross-section which is time- and placedependent. The validity of the equations is restricted to the suppositions we made to construct the equations (1) to (4) inclusive. Some of these suppositions are: a plane wave approach is sufficiently accurate; the changes of the tube's cross-section are relatively small; the acceleration $\frac{dv}{dt}$ can be replaced by the local acceleration $\frac{\partial v}{\partial t}$; the relation between the pressure p and the density ρ is a linear one and the medium inside the tube is rotation free.

In general, it is impossible to construct an analytical solution of (5) and (6). There fore, a numerical approach seems to be a necessary way to describe the acoustical behaviour of the tube.

The special case of a tube with a cross-section independent of time, gives to (5) and (6) the same mathematical form. In the following, we restrict ourselves to the description of the velocity potential. Equation (5) is reduced to

$$(7) \qquad \frac{\partial}{\partial x}\left(\sigma \frac{\partial \Phi}{\partial x}\right) = \frac{\sigma}{c^2}\frac{\partial^2 \Phi}{\partial t^2} ,$$

the equation of the inhomogeneous vibrating string or the so-called horn equation of Webster. Equation (7) offers the possibility to describe the acoustical properties of the vocal tract. Let π denote the length of the vocal tract. Assuming that the tube is ideally open on the mouth-side and ideally closed on the side of the vocal cords, then we have to solve the following problem:



$$(8) \quad \begin{aligned} &\Phi_{xx} + (\ln\sigma)_x \Phi_x = \frac{1}{c^2}\Phi_{tt} \\ &\Phi_x(0,t) = 0 \ , \quad \Phi(\pi, t) = 0 \ . \end{aligned}$$

In the problem stated above, all that has been done is posing the question of the quasi steady-states possible inside the tube. Following Bernoulli, we first look for special solutions of the type $\Phi(x,t)=y(x)f(t)$. Substitution of such special solutions in (8) leads to:

(9) $y_{xx} + (\ln\sigma)_x y_x + \lambda y = 0$, with the boundary conditions

$y_x(0) = 0 \ , \ y(\pi) = 0$

and

(10) $f_{tt} + c^2\lambda f = 0$, in which λ is a still undetermined constant.

The solution of (10) is clearly:

$$f(t) = A \sin c\sqrt{\lambda}\, t + B \cos c\sqrt{\lambda}\, t \ .$$

The constants A and B can be determined from initial conditions to be posited.

Substitution of a new independent variable $\xi = \int \frac{dx}{\sigma(x)}$

and replacing ξ by x, transforms (9) to:

(11) $y_{xx} + \lambda\sigma^2 y = 0$, the first normal form of Liouville.

The boundary conditions belonging to (11) are again homogeneous ones.

A second substitution $z=y\sqrt{\sigma}$ transforms (9) to

(12) $z_{xx} + (\lambda - g(x))z = 0$, in which

$$g(x) = \frac{(\sqrt{\sigma})_{xx}}{\sqrt{\sigma}},$$ the second normal form of Liouville,

together with homogeneous boundary conditions. Under ample mathematical conditions, it can be proved that (12) together with homogeneous boundary conditions can have non-vanishing solutions if and only if λ belongs to an infinite sequence $\lambda_0 < \lambda_1 < \lambda_2 < \cdots\cdots$.

Corresponding to each of these, there are solutions z_0, z_1, z_2, ..., and they satisfy the integral relationships

$$\int_0^{\pi} z_n z_m \, dx = 0, \quad n \neq m.$$

What is more, the solutions $\{z_n\}$ form a complete system. Because (12), together with the boundary conditions, is equivalent to (9), the above assertion is valid to (9). If we call the solution of (9) belonging to one λ_n the eigen-function y_n, then we construct as a special solution of (8): $\Phi_n(x,t) = y_n(x) f_n(t)$

in which $f_n(t) = A_n \sin c\sqrt{\lambda_n}\, t + B_n \cos c\sqrt{\lambda_n}\, t.$

Now, the general solution to (8) is the linear composition;

$$\Phi(x,t) = \sum_{0}^{\infty} C_n \Phi_n(x,t) \text{ .}$$

To solve equation (9) for any given cross-section $\sigma(x)$, we need numerical techniques. However, there are some special cases which lead to analytical solutions. Three of them are: $\sigma=\sigma_o e^{mx}$, the exponential horn. (9) reduces to

$$y_{xx} + m y_x + \lambda y = 0 \text{ ,}$$

an ordinary differential equation with constant coefficients, which can be easily solved. Now let us assume, $\sigma=\sigma_o e^{-ax^2}$. From equation (9) we obtain

$$(13) \qquad y_{xx} - 2a y_x + \lambda y = 0 \text{ .}$$

Substitution of $z=ax^2$ and $\lambda=2a\nu$ in (13) leads rapidly to the confluent-hypergeometric equation:

$$(14) \qquad z y_{zz} + \left(\tfrac{1}{2} - z\right) y_z + \tfrac{\nu}{2} y = 0 \text{ .}$$

Therefore, the solution of (13) can be written in terms of confluent-hypergeometric functions. And further, that $\sigma=\sigma_o x^n$, in which n is an integer. We find after substitution in (9):

$$(15) \qquad y_{xx} + \frac{n}{x} y_x + \lambda y = 0 \text{ .}$$

Let $y = z / p^{\frac{n-1}{2}}$ and $p = x\sqrt{\lambda}$. Equation (15) transforms to Bessel's equation:

$$(16) \qquad p^2 z_{pp} + p z_p + \left(p^2 - \left[\frac{n-1}{2}\right]^2\right) z = 0$$

If $\frac{n-1}{2}$ is not an integer, then (15) has

$$J_{\pm\frac{n-1}{2}}(x\sqrt{\lambda})\Big/(x\sqrt{\lambda})^{\frac{n-1}{2}}$$ as independent solutions.

However, if $\frac{n-1}{2}$ is an integer, then (15) has besides

$$J_{\frac{n-1}{2}}(x\sqrt{\lambda})\Big/(x\sqrt{\lambda})^{\frac{n-1}{2}}$$ as second independent solution:

$$Y_{\frac{n-1}{2}}(x\sqrt{\lambda})\Big/(x\sqrt{\lambda})^{\frac{n-1}{2}},$$ in which Y denotes the function of Neumann.

As an example we choose $\sigma=\sigma_0 x^2$, the conic tube, with a length of 20 cm. For the sake of simplicity, we assume the tube to be closed at x = 2 and open at x = 22. Then problem (8) is reduced to:

$$\Phi_{xx} + \frac{2}{x}\Phi_x = \frac{1}{c^2}\Phi_{tt}$$ with

$$\Phi_x(2,t) = 0 \, , \; \Phi(22,t) = 0 \, .$$

Using the method of Bernoulli, we find for the placedependent part of the special solutions $\Phi(x,t)=y(x)f(t)$:

$$y_{xx} + \frac{2}{x}y_x + \lambda y = 0 \, ,$$ with boundary conditions:

(17) $$y_x(2) = 0 \, , \; y(22) = 0 \, .$$

A solution to (17) appears to be

(18) $$y = C\,\frac{\sin\sqrt{\lambda}\,x}{x} + D\,\frac{\cos\sqrt{\lambda}\,x}{x} \, .$$

(18) must satisfy the boundary conditions at x = 2 and x = 22, resulting in

(19) $\operatorname{tg} 20\sqrt{\lambda} = -2\sqrt{\lambda}$ and

(20) $D = -C \operatorname{tg} 22\sqrt{\lambda}$.

Solving (19) in a graphic manner, we obtain:

(21)
$\sqrt{\lambda_1}=0.143$
$\sqrt{\lambda_2}=0.288$
$\sqrt{\lambda_3}=0.436$
$\sqrt{\lambda_4}=0.582$

Using (20), we find the eigen-function y_n:

$$y_n(x) = C_n\left\{\frac{\sin\sqrt{\lambda_n}\,x}{x} - \operatorname{tg} 22\sqrt{\lambda_n}\,\frac{\cos\sqrt{\lambda_n}\,x}{x}\right\}$$

Therefore, the special solutions for which we sought are of the type:

$y_n(x)\, f_n(t)$ with

$$f_n(t) = A_n \sin c\sqrt{\lambda_n}\,t + B_n \cos c\sqrt{\lambda_n}\,t$$

From this result, the general solution of the stated example follows immediately:

$$\Phi(x, t) = \sum_{1}^{\infty} y_n(x)\, f_n(t)$$

If we wish to express $f_n(t)$ in terms of the commonly used frequency scale, thus:

$$f_n(t) = A_n \sin 2\pi f_n t + B_n \cos 2\pi f_n t,$$

the relation between a λ_n and f_n is given by (22).

(22) $\lambda_n = \left(\frac{2\pi f_n}{c}\right)^2$

Let $c = 35000\ \mathrm{cm}/_{\mathrm{sec}}$. We find from (21) and (22)

f1= 795 Hz

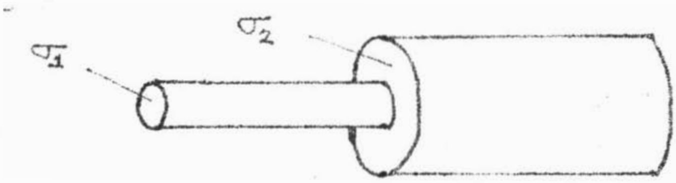
f2=1600 Hz

f3=2406 Hz

f4=3233 Hz

Suppose, we know the analytic solution corresponding to a certain given variable cross-section $\sigma(x)$ of the tube. This enables us to approximate the vocal tract with a chain of tubes (see. n-tube program below) of which the cross-sections are not necessarily a constant.

Laplace transformation gives the possibility of determining the influence of an impulse of the vocal-cords on the eigen-vibrations of the vocal tract. We illustrate this technique by means of the twin-tube model with a given vocal-cord impulse f(t). In its simplest form, the mathematical model assumes the form below.



(23)
$$\Phi^{(1)}_{xx} = \frac{1}{c^2}\Phi^{(1)}_{tt}\ ,\quad 0 < x < a$$
$$\Phi^{(2)}_{xx} = \frac{1}{c^2}\Phi^{(2)}_{tt}\ ,\quad a < x < \ell$$

with boundary conditions:

(24)
$$\Phi^{(1)}_{x}(0, t) = f(t)$$ the given impulse of the vocal-cords and

$$\Phi^{(2)}(\ell, t) = 0.$$ At x = a, the following continuity conditions

(25) $\Phi^{(1)} = \Phi^{(2)}, \sigma_1 \Phi_x^{(1)} = \sigma_2 \Phi_x^{(2)}$ are in force.

Let $u^{(1),(2)} = \int_0^\infty e^{-st} \Phi^{(1),(2)}(x,t)\,dt$, the Laplace transforms of the velocity potentials $\Phi^{(1)}$ and $\Phi^{(2)}$, then (23) to (25) inclusive transforms to

$$(26)\quad \begin{aligned} &u_{xx}^{(1)} - s^2 u^{(1)} = 0, \quad 0 < x < a \\ &u_{xx}^{(2)} - s^2 u^{(2)} = 0, \quad a < x < l, \end{aligned}$$

in which for the sake of simplicity another timescale is chosen, so that $c = 1$,

$$(27)\quad \begin{aligned} &u_x^{(1)}(0,s) = \bar{f}(s), \\ &u^{(2)}(l,s) = 0 \end{aligned}$$

with $\bar{f}(s) = \mathcal{L}\{f(t)\}$ and

(28) $u^{(1)} = u^{(2)}, \sigma_1 u_x^{(1)} = \sigma_2 u_x^{(2)}$ as the transformation of the continuity conditions at $x = a$.

The solution of (26) adapted to the conditions (27) appears to be

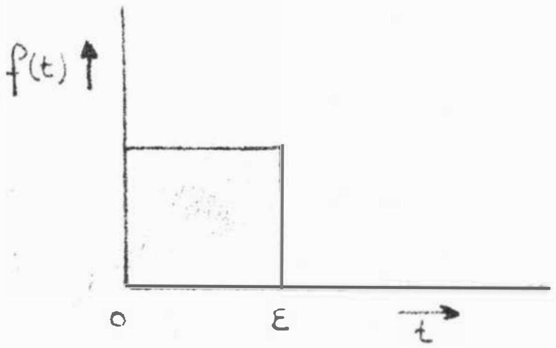
$$(29)\quad \begin{aligned} &u^{(1)} = \frac{\bar{f}(s)}{s} \sinh sx + A \cosh sx \\ &u^{(2)} = B\,(\sinh sx - \operatorname{tgh} sl \cosh sx) \end{aligned}$$

The constants A and B can be determined from the conditions (28). We obtain:

$$A = -\frac{\bar{f}(s)}{s}\,\frac{\sigma_1 \operatorname{tgh} sb + \sigma_2 \operatorname{tgh} sa}{\sigma_2 + \sigma_1 \operatorname{tgh} sa \,.\, \operatorname{tgh} sb} \quad \text{and}$$

$$B = \frac{\bar{f}(s)}{s}\,\frac{\sigma_1 + \sigma_1 \operatorname{tgh} sa \operatorname{tgh} sb}{\sigma_2 + \sigma_1 \operatorname{tgh} sa \operatorname{tgh} sb}, \quad b = l - a.$$

As an example of an impulse of the vocal-cords, we choose a block-shaped time-function with a variable duration



$$f(t) = \delta(t) - \delta(t - \varepsilon),$$

then $$\bar{f}(s) = \frac{1 - e^{-\varepsilon s}}{s}$$

Substition of $\bar{f}(s)$ and the constants A and B into (29), makes it possible to determine the sought for velocity potentials $\Phi^{(1)}$ and $\Phi^{(2)}$ with the aid of the complex inversion formula

$$\Phi^{(1),(2)}(x,t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\, u^{(1),(2)}(x,s)\, ds$$

Then, the vibrations caused by the impulse of the vocal-cords are known.

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